ISI B. Math.
Physics I
[redacted] Semetral Exam
Total Marks: 100

Answer any five questions. All questions carry equal marks.

1. Consider a charged particle of mass m and charge q entering a uniform constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{k}}$. The force on the charged particle is given by

$$\mathbf{F} = \frac{q}{c}(\mathbf{v} \times \mathbf{B})$$

where c is the speed of light.
a) Show that the kinetic energy of the particle is a constant of motion. (5)
b) Given that at time $t = 0$, the particle starts from the origin with $\dot{x} = 0, \dot{y} = \dot{y}_0, \dot{z} = \dot{z}_0$, find the subsequent motion of the particle and make a rough sketch of its trajectory. How will the trajectory be affected by increasing the magnetic field? (15)

2. (i) A boat with initial velocity v_0 is slowed down by a frictional force

$$F = -be^{\alpha v}$$

(a) Find its motion. (7)
(b) Find the time and distance required to stop. (7)
(ii) Another boat is slowed down by a frictional force that decreases according to the formula

$$v = C(t - t_1)^2,$$

where C is a constant and t_1 is the time at which it stops. Find the force $F(v)$. (6)

3. (a) The balance wheel of a watch consists of a ring of mass M of radius a with spokes of negligible mass. The hairspring exerts a restoring torque $N_z = -k\theta$. Find $\theta(t)$ if the wheel is rotated through an angle θ_0 and then released. (6)
(b) Consider a dumbell : an iron rod with two identical spherical masses attached to the ends . How many degrees of freedom does this body have?

Is it a rigid body? Why is the number of degrees of freedom of this body less than 6, the generic number for a rigid body ?(3)

(c) A circular sheet of aluminium of radius A has a concentric hole of radius a punched out from it. Where is the centre of mass of this system located ?(4)

(d) If the above aluminium disc (with the hole punched out) has a mass M, find its moment of inertia about an axis through the centre of mass perpendicular to the plane of the sheet. (7)

4. (a) Consider a rectangular block of elastic material that is stretched along its length such that its length l is increased by Δl. During this process, appropriate forces are applied to the block to ensure that there is no contraction of the block in the directions perpendicular to the direction in which the block is stretched. Show that

$$\frac{F}{A} = \frac{1-\sigma}{(1+\sigma)(1-2\sigma)} Y \frac{\Delta l}{l}$$

where F is the magnitude of the stretching force, A is the cross-sectional area perpendicular to the force, Y is the Young's modulus and σ is Poisson's ratio for the material. Is the factor multiplying the strain greater than or less than 1 ? Explain. (10)

(b) If the torsional rigidity (torque per unit twist) of a wire is given by c, and a body of moment of inertia I suspended from the wire undergoes small torsional oscillations, show that the period of small oscillations T is given by $2\pi\sqrt{\frac{I}{c}}$. Consider the following experimental arrangement . One end of a wire is fixed vertically to a rigid support and the other end is fixed to the centre of a metallic disc. Two known masses (m each) are placed symmetrically along the diameter at a given distance and the disc + mass system is made to undergo small torsional oscillations. Show that the rigidity modulus μ is given by

$$\mu = \frac{16\pi ml(d_2^2 - d_1^2)}{a^4(T_2^2 - T_1^2)}$$

where T_2 and T_1 are the time periods of oscillation when the masses are placed at distances d_2 and d_1 respectively. l and a are the length and radius of the wire respectively and $c = \frac{\pi\mu a^4}{2l}$. This is an experimental arrangement to determine the modulus of rigidity of a wire. (10)

5. (a) Use the technique of calculus of variations to show that the curve of shortest length between two fixed points in a plane is a straight line. (7)
(b) Consider a massless, frictionless pulley with a mass M_1 hanging at one end and a mass M_2 hanging from the other end . Write down the Lagrangian $L(x, \dot{x})$ for the system and find the acceleration of the masses by applying the Euler-Lagrange equation to the Lagrangian. (6)
(c) Consider a free particle of mass m moving in one dimension with respect to a certain inertial frame K. Now consider another inertial frame K' moving with a constant velocity v with respect to K. Show that the Lagrangian L' in the frame K' differs from the Lagrangian L in K by a total time derivative of a function of coordinates and time. (7)

6. a) Use the continuity equation to show that the flow defined by the velocity field

$$(2t + 2x + 2y)\hat{\mathbf{i}} + (t - y - z)\hat{\mathbf{j}} + (t + x - z)\hat{\mathbf{k}}$$

is incompressible. (6)
b) Euler's equation for an ideal fluid with velocity $\mathbf{v}$, pressure p and density ρ flowing under the influence of an external force with potential ϕ is given by

$$\frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} - \nabla \phi$$

Show that this reduces to

$$\frac{\partial \mathbf{\Omega}}{\partial t} + \nabla \times (\mathbf{\Omega} \times \mathbf{v}) = \mathbf{0}$$

for incompressible flow, where the vorticity $\mathbf{\Omega} = \nabla \times \mathbf{v}$ (8)
(c) Suppose we have water flowing out of a hole at the bottom of a tank as shown in the figure. Find the velocity v_{out} of the water flowing out of the hole in terms of the depth h of the hole using Bernoulli's principle.Assume the diameter of the tank is so large that we can neglect the drop in the liquid level. (6)

Some formulas that may be useful:

$$(\mathbf{A} \cdot \nabla)\mathbf{A} = (\nabla \times \mathbf{A}) \times \mathbf{A} + \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$$